a uniform pressure p_0^+ acts on the external contour and $\varepsilon = 1/s$, these coefficients are

$$K_I + iK_{II} = \sqrt{\pi c}\, p_0^+ [G_1(\theta_0, \varepsilon) + iG_2(\theta_0, \varepsilon)]$$

The values of $G_1(\theta_0, \varepsilon)$ and $G_2(\theta_0, \varepsilon)$ are presented in Table 1.

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ON THE DETERMINATION OF FORCES OF CONSTRAINT REACTION

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DO SHAN'
(Hanoi)
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The equations of motion of mechanical systems with multipliers are reduced to the form enabling the separation of these equations into two groups, the first group describing the motions of the system, and the second group defining the multipliers. Each multiplier is determined independently of the remaining multipliers, and this makes it easy to assess the dynamic effect of each constraint on the system. On the basis of this approach, we study the following problems: determination of the constraint reactions [1], study of the motion of controlled systems with prescribed constraints [2, 3] and utilization of the method of nonholonomic mechanical systems in the case when the first integrals exist [4].

1. Equations of motion of a system with multipliers. We consider a system the position of which is defined in terms of the generalized coordinates q_i ($i = 1, 2, \ldots, n$). We assume that the system is restricted by ideal, nonholonomic, second order nonlinear constraints of the form

$$f_\alpha(t, q_i, q_i^{\cdot}, q_i^{\cdot\cdot}) = 0, \quad i = 1, 2, \ldots, n; \quad \alpha = 1, 2, \ldots, \Delta \qquad (1.1)$$

where
$$D(f_1, f_2, \ldots, f_s)/D(\ddot{q}_{p+1}, \ldots, \ddot{q}_n) \neq 0, \quad p = n - s \tag{1.2}$$

Equations of motion of the system can be written in the form [5]

$$\frac{\partial S}{\partial \ddot{q}_i} = Q_i + \sum_{\alpha=1}^{s} \lambda_\alpha \frac{\partial f_\alpha}{\partial \ddot{q}_i}, \quad i = 1, 2, \ldots, n \tag{1.3}$$

where λ_α are the undetermined multipliers characterizing the dynamic effect of the constraints on the mechanical system. Since Eqs. (1.3) appear unsuitable, as they are, for the direct determination of the multipliers λ_α, we transform them by introducing new variables $\ddot{q}_1, \ddot{q}_2, \ldots, \ddot{q}_p, \ddot{u}_1, \ddot{u}_2, \ldots, \ddot{u}_s$ to replace $\ddot{q}_1, \ddot{q}_2, \ldots, \ddot{q}_n$. Here $\ddot{u}_\alpha = f_\alpha(t, q_i, \dot{q}_i, \ddot{q}_i)$.

Taking into account the condition (1.2), we have

$$\ddot{q}_h = \ddot{q}_h(t, q_i\, \dot{q}_i, \ddot{q}_\nu, \ddot{u}_\alpha), \quad h = p+1, \ldots, n; \quad \nu = 1, 2, \ldots, p; \quad \alpha = 1, 2, \ldots, s$$

Thus we can express the new variables in terms of the old ones, and vice versa. We note that the equations of constraints (1.1) now become

$$\ddot{u}_\alpha = 0, \quad \alpha = 1, 2, \ldots, s$$

Using the new variables we can write the equations of motion of the system in the following form [5]:

$$\frac{\partial S^\circ}{\partial \ddot{q}_\nu} = Q_\nu + \sum_{h=p+1}^{n} Q_h \frac{\partial \ddot{q}_h^\circ}{\partial \ddot{q}_\nu}, \quad \nu = 1, 2 \ldots, \quad p = n - s \tag{1.4}$$

$$\left(\frac{\partial S}{\partial \ddot{u}_\alpha}\right)^\circ = \sum_{h=p+1}^{n} Q_h \left(\frac{\partial \ddot{q}_h}{\partial \ddot{u}_\alpha}\right)^\circ + \lambda_\alpha, \quad \alpha = 1, 2, \ldots, s \tag{1.5}$$

$$S = S(t, q_i, \dot{q}_i, \ddot{q}_\nu, \ddot{u}_\alpha), \quad \Phi^\circ = \Phi\Big|_{\sum \ddot{u}_\alpha^2 = 0}$$

The first group of Eqs. (1.4) describes the motion of the system, and the second group (1.5) makes it possible to determine the multipliers independently from each other.

Notes. 1°. The above equations can obviously be applied to the systems with the usual first order nonholonomic constraints.

2°. Since for the holonomic systems the constraint equations have the form

$$F_\alpha(t, q_i) = 0, \quad i = 1, 2, \ldots, n; \quad \alpha = 1, 2, \ldots, s$$

we introduce the following variables: $q_1, q_2, \ldots, q_p, u_1, u_2, \ldots, u_s$ with $u_\alpha = F_\alpha(t, q_i)$. Then Eqs. (1.4) and (1.5) will assume the form

$$\frac{d}{dt}\frac{\partial T^\circ}{\partial \dot{q}_\nu} - \frac{\partial T^\circ}{\partial q_\nu} = Q_\nu^\circ + \sum_{h=p+1}^{n} Q_h^\circ \frac{\partial q_h^\circ}{\partial q_\nu}, \quad \nu = 1, 2, \ldots, \quad p = n - s \tag{1.6}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_\alpha}\right)^\circ - \left(\frac{\partial T}{\partial u_\alpha}\right)^\circ = \sum_{h=p+1}^{n} Q_h^\circ \left(\frac{\partial q_h}{\partial u_\alpha}\right)^\circ + \lambda_\alpha, \quad \alpha = 1, 2, \ldots, s \tag{1.7}$$

$$T = T(t, q_\nu, \dot{q}_\nu\ u_\alpha, \dot{u}_\alpha), \quad \Phi^\circ = \Phi\Big|_{\sum \dot{u}_\alpha^2 = \sum u_\alpha^2 = 0}$$

Example 1. Let us consider the motion of a motorcar in accordance with a simplified scheme (see [6]). Following [6], we write the expression for the energy of acceleration in the form

$$2S = M\,(x_c^{\cdot\cdot 2} + y_c^{\cdot\cdot 2}) + J_c \varphi^{\cdot\cdot 2} + \ldots$$

where the terms which have not been written out contain no $x_c^{\cdot\cdot}$, $y_c^{\cdot\cdot}$ or $\varphi^{\cdot\cdot}$. The generalized forces referred to the generalized coordinates x_c, y_c and φ are $Q = F\cos\varphi$, $Q_2 = F\sin\varphi$ and $Q_3 = 0$ [6].

Let us now introduce the variables φ, u_1 and u_2

$$u_1^{\cdot} = x_c^{\cdot}\sin\varphi - y_c^{\cdot}\cos\varphi + b\varphi^{\cdot}, \quad u_2^{\cdot} = x_c^{\cdot}\sin(\varphi+\theta) - y_c^{\cdot}\cos(\varphi+\theta) - a\cos\theta\,\varphi^{\cdot}$$

($u_1^{\cdot} = 0$ and $u_2^{\cdot} = 0$ represent the equations of nonholonomic constraints [6]). Differentiating the above relations with respect to time and solving the resulting equations for $x_c^{\cdot\cdot}$ and $y_c^{\cdot\cdot}$, we have

$$x_c^{\cdot\cdot} = \frac{1}{\sin\theta}[A\cos(\varphi+\theta) + B\cos\varphi], \quad y_c^{\cdot\cdot} = \tag{1.8}$$

$$\frac{1}{\sin\theta}[B\sin\varphi + A\sin(\varphi+\theta)], \quad A = b\varphi^{\cdot\cdot} - u_1^{\cdot\cdot} + l\,\mathrm{ctg}\,\theta\,\varphi^{\cdot 2}$$

$$B = u_2^{\cdot\cdot} + a\cos\theta\,\varphi^{\cdot\cdot} - \frac{l\cos^2\theta + b\sin^2\theta}{\sin\theta}\,\varphi^{\cdot 2} - \frac{l}{\sin\theta}\,\varphi\theta^{\cdot\cdot}, \quad l = a + b$$

Using Eq. (1.4), we now obtain the equation of motion of the system

$$J_c\varphi^{\cdot\cdot} + \frac{M}{\sin\theta}\{x_c^{\cdot\cdot}[b\cos(\varphi+\theta) + a\cos\theta\cos\varphi] + \tag{1.9}$$

$$y_c^{\cdot\cdot}[a\cos\theta\sin\varphi + b\sin(\varphi+\theta)]\} = 0$$

To find the dynamic effects of the constraints we use (1.5) to obtain

$$\frac{M}{\sin\theta}[x_c^{\cdot\cdot}\cos(\varphi+\theta) + y_c^{\cdot\cdot}\sin(\varphi+\theta)] = F\,\mathrm{ctg}\,\theta - \lambda_1 \tag{1.10}$$

$$M / \sin\theta\,(x_c^{\cdot\cdot}\cos\varphi + y_c^{\cdot\cdot}\sin\varphi) = F + \lambda_2$$

In (1.10) we must replace $x_c^{\cdot\cdot}$ and $y_c^{\cdot\cdot}$ by the corresponding expressions given in (1.8). After simple transformations we obtain the expressions for $\varphi^{\cdot\cdot}$, λ_1 and λ_2 which agree with those obtained in [6].

2. On the motion of controlled mechanical systems with prescribed constraints. Let us allow real, ideal constraints (2.1) and prescribed constraints [2, 3] (2.2) be imposed on a controlled mechanical system

$$f_\alpha(t, q_i, q_i^{\cdot}, q_i^{\cdot\cdot}, \xi_j) = 0, \quad i = 1, 2, \ldots, n; \quad \alpha = 1, 2, \ldots, s; \quad j = 1, 2, \ldots, k \tag{2.1}$$

$$g_\beta(t, q_i, q_i^{\cdot}, q_i^{\cdot\cdot}, \xi_j) = 0, \quad \beta = 1, 2, \ldots, r \tag{2.2}$$

where ξ_j are the control parameters of the system. Let us also assume, that

$$\frac{D(f_1, f_2, \ldots, f_s, g_1, g_2, \ldots, g_r)}{D(q_{p+1}^{\cdot\cdot}, q_{p+2}^{\cdot\cdot}, \ldots, q_n^{\cdot\cdot})} \neq 0, \quad p = n - (s + r)$$

The prescribed constraints can be regarded as constraints the reactions of which are equal to zero.

Let us determine the reactions of the prescribed constraints and equate them to zero. We introduce the variables $q_1^{\cdot\cdot}$, $q_2^{\cdot\cdot}$, $\ldots$, $q_p^{\cdot\cdot}$, $u_1^{\cdot\cdot}$, $u_2^{\cdot\cdot}$, $\ldots$, $u_s^{\cdot\cdot}$, $v_1^{\cdot\cdot}$, $v_2^{\cdot\cdot}$, $\ldots$, $v_r^{\cdot\cdot}$, and

$$u_\alpha^{\cdot\cdot}=f_\alpha(t,q_i,q_i^{\cdot},q_i^{\cdot\cdot},\xi_j),\quad v^{\cdot\cdot}_\beta=g_\beta(t,q_i,q_i^{\cdot},q_i^{\cdot\cdot},\xi_j),$$
$$\alpha=1,2,\ldots,s;\quad \beta=1,2,\ldots,r$$

Using Eqs. (1. 1), we obtain the dynamic conditions

$$\left(\frac{\partial S}{\partial v_\beta^{\cdot\cdot}}\right)=\sum_{h=p+1}^{n}Q_h\left(\frac{\partial q_h^{\cdot\cdot}}{\partial v_\beta^{\cdot\cdot}}\right)^\circ,\quad \beta=1,2,\ldots,r \tag{2.3}$$

The equations of motion of the reduced system now assume the form

$$\frac{\partial S^\circ}{\partial q_\nu^{\cdot\cdot}}=Q_\nu+\sum_{h=p+1}^{n}Q_h\frac{\partial q_h^{\cdot\cdot\circ}}{\partial q_\nu^{\cdot\cdot}},\quad \nu=1,2,\ldots,p \tag{2.4}$$

and equations determining the reactions of the real constraints, become

$$\left(\frac{\partial S}{\partial u_\alpha^{\cdot\cdot}}\right)^\circ=\sum_{h=p+1}^{n}Q_h\left(\frac{\partial q_h^{\cdot\cdot}}{\partial u_\alpha^{\cdot\cdot}}\right)^\circ+\lambda_\alpha,\quad \alpha=1,2,\ldots,s \tag{2.5}$$

Thus the motion of a controlled system with prescribed constraints is completely described by Eqs. (2. 3) and (2. 4). Equations (2. 4) represent the equations of motion of the system in question under the assumption that the prescribed constraints are regarded as real constraints, while Eqs. (2. 3) describe the conditions which enable us to reduce the prescribed constraints to the real ones. Equations (2. 5) give the reactions of the real constraints. Thus if the reduced system has p degrees of freedom, then the number of equations describing the motions of the system is $p+r$ (where r is the number of equations of prescribed constraints). Equations (2. 1), (2. 3) and (2. 4) together form a complete system of equations of the problem of motion of a controlled mechanical system with prescribed constraints.

Note. If we impose on holonomic mechanical systems the prescribed holonomic constraints (2. 6) and real holonomic constraints (2. 7)

$$g_\beta(t,q_i,\xi_j)=0,\quad i=1,2,\ldots,n;\quad \beta=1,2,\ldots,r;\quad j=1,2,\ldots,k \tag{2.6}$$

$$f_\alpha(t,q_i,\xi_j)=0,\quad \alpha=1,2,\ldots,s \tag{2.7}$$

replace the variables $q_1, q_2, \ldots, q_n$ by the variables $q_1, q_2, \ldots, q_p, u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_r$ and

$$u_\alpha=f_\alpha(t,q_i,\xi_j),\quad v_\beta=g_\beta(t,q_i,\xi_j)$$

then the dynamic conditions will assume the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v_\beta^{\cdot}}\right)^\circ-\left(\frac{\partial T}{\partial v_\beta}\right)^\circ=\sum_{h=p+1}^{n}Q_h^\circ\left(\frac{\partial q_h}{\partial v_\beta}\right)^\circ,\quad \beta=1,2,\ldots,r$$

and the equations of motion of the system will become

$$\frac{d}{dt}\frac{\partial T^\circ}{\partial q_\nu^{\cdot}}-\frac{\partial T^\circ}{\partial q_\nu}=Q_\nu^\circ+\sum_{h=p+1}^{n}Q_h^\circ\frac{\partial q_h^\circ}{\partial q_\nu},\quad \nu=1,2,\ldots,p$$

while the equations describing the reactions of the real constraints will have the form

$$\frac{d}{dt}\left(\frac{\partial T^\circ}{\partial u_\alpha^{\cdot}}\right)^\circ-\left(\frac{\partial T^\circ}{\partial u_\alpha}\right)^\circ=\sum_{h=p+1}^{n}Q_h^\circ\left(\frac{\partial q_h}{\partial u_\alpha}\right)^\circ,\quad \alpha=1,2,\ldots,s$$

Example 2. The Appell problem [2, 3]. A material plane P can slide translationally on a stationary horizontal plane Oxy. A sphere of radius R can roll without sliding on the plane P. The motion on the plane P is regulated automatically in such a way, that the center of the sphere rotates uniformly about the Oz-axis with angular velocity ω.

Let us write the equations of the problem. We denote by ξ and η the coordinates of the center of the sphere, by p, q and r the components of the instantaneous angular velocity of the sphere, and by u, v the coordinates of some point on the plane. Then the energy of acceleration of the sphere is

$$2S = M(\xi^{\cdot\cdot 2} + \eta^{\cdot\cdot 2}) + {}^2/_5 MR^2(p^{\cdot 2} + q^{\cdot 2} + r^{\cdot 2})$$

The equations of the real constraints are $\xi^{\cdot} - qR - u^{\cdot} = 0$, $\eta^{\cdot} + pR - v^{\cdot} = 0$, and the equations of the prescribed constraints are $\xi^{\cdot} + \omega\eta = 0$, $\eta^{\cdot} - \omega\xi = 0$. Introducing the variables r, u_1, u_2, v_1 and v_2 in which

$$u_1^{\cdot} = \xi^{\cdot} - qR - u^{\cdot}, \quad u_2^{\cdot} = \eta^{\cdot} + pR - v^{\cdot}, \quad v_1^{\cdot} = \xi^{\cdot} + \omega\eta, \quad v_2^{\cdot} = \eta^{\cdot} - \omega\xi$$

we find

$$\xi^{\cdot\cdot} = v_1^{\cdot\cdot} - \omega v_2^{\cdot} - \omega^2\xi, \quad \eta^{\cdot\cdot} = v_2^{\cdot\cdot} + \omega v_1^{\cdot} - \omega^2\eta$$

$$q^{\cdot} = \frac{1}{R}(v_1^{\cdot\cdot} - u_1^{\cdot\cdot} - \omega v_r^{\cdot} - \omega^2\xi - u^{\cdot\cdot}), \; p^{\cdot} = \frac{1}{R}(u_2^{\cdot\cdot} - v_2^{\cdot\cdot} + \omega v_1^{\cdot} + \omega^2\eta + v^{\cdot\cdot})$$

and the energy of acceleration assumes the form

$$2S = M[(v_1^{\cdot\cdot} - \omega v_2^{\cdot} - \omega^2\xi)^2 + (v_2^{\cdot\cdot} + \omega v_1^{\cdot} - \omega^2\eta)^2] +$$
$$^2/_5 M[(v_1^{\cdot\cdot} - u_1^{\cdot\cdot} - \omega v_2^{\cdot} - \omega^2\xi - u^{\cdot\cdot})^2 + (u_2^{\cdot\cdot} - v_2^{\cdot\cdot} - \omega v_1^{\cdot} + \omega^2\eta + v^{\cdot\cdot})^2 + R^2 r^2]$$

The generalized forces expressed in terms of the generalized coordinates, are equal to zero. In accordance with Eqs. (2.3), the dynamic conditions are

$$7\omega^2\xi + 2u^{\cdot\cdot} = 0, \quad 3\omega^2\eta - 2v^{\cdot\cdot} = 0$$

and (2.4) yields the equations of motion $r^{\cdot} = 0$ of the reduced system.

To find the reactions of the real constraints we use (2.5) and obtain $^2/_5 MRq^{\cdot} = -\lambda_1$, $^2/_5 MRp^{\cdot} = \lambda_2$.

It can easily be shown that the dynamic conditions can be written in the form

$$-5\omega^2\xi + 2Rq^{\cdot} = 0, \quad 2Rp^{\cdot} - 5\omega^2\eta = 0$$

Thus the system (2.6) is of the same order as the system (2.7) [3].

3. We consider the case when the integrals of dynamic equations represent the conditions of the nonholonomic constraints [3].

We assume that certain first integrals of motion of the system are known. The question arises whether these integrals can be regarded as the equations of nonholonomic constraints imposed on the system in question, and how to use them in constructing the equations. Applying the results obtained in Sect. 2 we obtain the answer very simply, since the first integrals are particular cases of the prescribed constraints. In other words, the first integrals can be regarded as the equations of nonholonomic constraints.

Let the motion of the system be subject to the nonholonomic constraint (1.1) and have the first integrals

$$g_\beta(t, q_i, q_i^{\cdot}) = C_\beta = \text{const}, \quad \beta = 1, 2, \ldots, r \tag{3.1}$$

The equations of motion of the system with constraints (1.1) will be

$$\frac{\partial S}{\partial q_i^{\cdot\cdot}} = Q_i + Q_i^*, \quad Q_i^* = \sum_{\alpha=1}^{s} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i^{\cdot\cdot}} \tag{3.2}$$

where Q_i^* denote the generalized forces of reaction of the constraints. Obviously, Eqs. (3.2) also have first integrals (3.1). Let us now regard them as the new constraints and find the corresponding Lagrange multipliers. As before, we introduce the variables

$$v_\beta^{\cdot\cdot} = \sum_{i=1}^{n} \frac{\partial g_\beta}{\partial q_i^{\cdot}} q_i^{\cdot\cdot} + \sum_{i=1}^{n} \frac{\partial g_\beta}{\partial q_i} q_i^{\cdot} + \frac{\partial g_\beta}{\partial t}$$

Utilizing (1.5) and noting that

$$\frac{\partial S}{\partial v_\beta^{\cdot\cdot}} = \sum_{i=1}^{n} \frac{\partial S}{\partial q_i^{\cdot\cdot}} \frac{\partial q_i^{\cdot\cdot}}{\partial v_\beta^{\cdot\cdot}}$$

we find the multipliers μ_β corresponding to the "new" constraints

$$\mu_\beta = \sum_{i=1}^{n} \left(\frac{\partial S}{\partial q_i^{\cdot\cdot}} - Q_i - Q_i^* \right)^\circ \left(\frac{\partial v_\beta^{\cdot\cdot}}{\partial q_i^{\cdot\cdot}} \right)^\circ$$

But along the trajectory of motion we have (3.2), therefore

$$\mu_\beta = 0, \quad \beta = 1, 2, \ldots, r$$

Example 3. We consider the case of a holonomic scleronomous system with ignorable coordinates q_α ($\alpha = m+1, \ldots, n$). The kinetic energy of such a system has the form

$$T = \frac{1}{2} \sum_{r, s=1}^{n} a_{rs}(q_1, q_2, \ldots, q_m) q_r^{\cdot} q_s^{\cdot}$$

In this case the cyclic integrals exist

$$p_\alpha = \partial T / \partial q_\alpha^{\cdot} = h_\alpha = \text{const}, \quad \alpha = m+1, \ldots, n \tag{3.3}$$

We can now regard the cyclic integrals as the equations of the nonholonomic constraints. Let us introduce the variables $q_1^{\cdot}, q_2^{\cdot}, \ldots, q_m^{\cdot}, v_{m+1}^{\cdot}, v_{m+2}^{\cdot}, \ldots, v_n^{\cdot}$, where $v_\beta^{\cdot} = p_\beta - h_\beta$, $\beta = m+1, \ldots, n$.

First, we shall show that the dynamic conditions are satisfied identically. In the present case these conditions have the form

$$\left(\frac{\partial S}{\partial v_\beta^{\cdot\cdot}} \right)^\circ = \sum_{\alpha=m+1}^{n} Q_\alpha \left(\frac{\partial q_\alpha^{\cdot\cdot}}{\partial v_\beta^{\cdot\cdot}} \right)^\circ$$

It can easily be proved that

$$\left(\frac{\partial S}{\partial v_\beta^{\cdot\cdot}} \right)^\circ = \sum_{\alpha=m+1}^{n} \frac{d}{dt} \left(\frac{\partial T}{\partial q_\alpha^{\cdot}} \right)^\circ \left(\frac{\partial q_\alpha^{\cdot\cdot}}{\partial v_\beta^{\cdot\cdot}} \right)^\circ \equiv 0$$

Moreover, we have $Q_\alpha \equiv 0$ since q_α are ignorable coordinates. Therefore the dynamic conditions are satisfied identically and the system is reducible. In accordance with [7], the constraint equations (3.3) yield

$$q_\alpha^{\cdot} = \sum_{i=1}^{m} B_{\alpha i} q_i^{\cdot} + B_\alpha, \; B_\alpha = \sum_{\beta=m+1}^{n} b_{\alpha\beta} h_\beta, \; B_{\alpha i} = -\sum_{\beta=m+1}^{n} b_{\alpha\beta} a_{\beta i} \tag{3.4}$$

$$\alpha, \beta = m+1, \ldots, n, \quad i = 1, \ldots, m$$

To construct the equations of motion of the reduced system, we shall use the Voronets equation [6] for the cyclic systems

$$\frac{d}{dt}\frac{\partial T^{\bullet}}{\partial q_i^{\cdot}}-\frac{\partial T^{\circ}}{\partial q_i}=-\frac{\partial \Pi}{\partial q_i}-\sum_{\alpha=m+1}^{n}\sum_{j=1}^{m}h_\alpha\left(\frac{\partial B_{\alpha j}}{\partial q_i}-\frac{d}{dt}B_{\alpha j}\right)-\sum_{\alpha=m+1}^{n}h_\alpha\frac{\partial B_\alpha}{\partial q_i} \qquad (3.5)$$

Here [1]

$$T^{\bullet}=T^{*}+\frac{1}{2}\sum_{\alpha,\,\beta=m+1}^{n}b_{\alpha\beta}h_\alpha h_\beta,\qquad T^{*}=\frac{1}{2}\sum_{i,\,j=1}^{m}a_{ij}^{*}q_i^{\cdot}q_j^{\cdot} \qquad (3.6)$$

T^* is the kinetic energy of the reduced system, while the coefficients a_{ij}^* and $b_{\alpha\beta}$ can be found as shown in [7].

Let us now set $L^* = T^* - V$, where V is the generalized potential defined by the equation [7]

$$V=\Pi^{*}+\frac{1}{2}\sum_{\alpha=m+1}^{n}\sum_{j=1}^{m}B_{\alpha j}h_\alpha q_j^{\cdot},\quad \Pi^{*}=\Pi+\frac{1}{2}\sum_{\alpha,\,\beta=m+1}^{n}b_{\alpha\beta}h_\alpha h_\beta$$

where Π^* is the Routh potential. Taking into account the relations (3.4) and (3.6), we write (3.5) in the form

$$\frac{d}{dt}\frac{\partial L^{*}}{\partial q_i^{\cdot}}-\frac{\partial L^{*}}{\partial q_i}=0,\qquad i=1,2,\ldots,m$$

The latter represent the equations of motion of the reduced system containing the generalized Lagrangian function [7].

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